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Transition to Stochasticity in a One-Dimensional Model of a Radiant Cavity

Giancarlo Benettin[1] and Luigi Galgani[2]

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We make a numerical study of the solutions of the equations of motion for the electromagnetic field in a one-dimensional model of a radiant cavity. Our main results are as follows: (1) There exist stochasticity thresholds such that below them one has ordered motions without energy exchanges, while chaotic motions with intense energy exchanges occur above them; (2) above thresholds there is a trend toward equipartition of energy (in time average) among the normal modes of the field, but this occurs in the sense of Boltzmann and Jeans, namely, with the higher frequencies requiring longer and longer times in order to be involved in the energy sharing.

KEY WORDS: Stochasticity threshold; blackbody.

1. INTRODUCTION

a. One of the most famous problems of classical statistical physics concerns the distribution of energy among the infinite oscillators (or normal modes) constituting the electromagnetic field in a cavity. The common attitude has always been to think that equipartition would obtain; this was inferred by analogy, because equipartition was believed to have been proven, on the basis of dynamics, for systems of a finite number of mechanical oscillators. On the other hand, this problem was recently reopened after the establishment of the Kolomogorov, Arnol'd, and Moser theorem, according to which a finite system of classical oscillators at low enough energy is in general nonergodic, so that equipartition of energy does not necessarily follow.

[1] Istituto di Fisica dell'Università e Gruppo Nazionale di Struttura della Materia del C.N.R., Padova, Italy.
[2] Istituto di Fisica e Istituto di Matematica dell'Università, Milano, Italy.

In fact, the typical situation occurring for finite systems of weakly coupled oscillators can be described in a first rough approximation, on the basis of several recent numerical computations, by saying that one has ordered motions (with an energy distribution strongly dependent on the initial conditions) at low enough energies, and chaotic motions (roughly leading to equipartition) at higher energies. And thus, still by analogy, one could then guess that a similar situation should obtain also for the electromagnetic field in a cavity.[1]

b. The problem is then to discuss, on the basis of dynamics, the open problem of the distribution of energy among the oscillators constituting the electromagnetic field in a cavity. This was first accomplished for a simple model, proposed by Bocchieri *et al.*,[2] in which one considers the electromagnetic field between two fixed infinite parallel mirrors; in order to avoid difficulties related to the structure of elementary charges, a coupling among the modes of the field is provided by a macroscopic object, precisely a uniformly charged infinite plane parallel to the mirrors, situated midway between them, which can translate along a given direction parallel to the mirrors. Such a model is actually one dimensional, and the equations of motion turn out to be linear. Consequently the system is integrable and does not lead to equipartition.

The interesting problem is then to see what happens when a nonlinearity is added. Thus in Refs. 3 and 4 the nonlinearity was provided by the addition of a mechanical restoring force acting on the charged plane; on the basis of numerical computations, indications were found of no qualitative change with respect to the linear case within a broad range of values of the parameters characterizing the model. On the other hand, a very interesting analytical result was found by Guarneri and Toscani,[5] who studied a variant of the model, where the charged plane, acted upon by a purely linear spring, is subjected to a stochastic perturbation simulating a thermal contact with a heat bath. In such a model they proved the existence of a trend toward equipartition of energy among the modes of the field, every mode having, however, a characteristic relaxation time toward equipartition, increasing with frequency, as proposed long ago by Boltzmann and Jeans.[6]

In the meantime, systems of coupled oscillators corresponding to models of mechanical type were intensively studied numerically, and the phenomenon of a transition to stochasticity was discussed in terms of the notion of an energy threshold for stochasticity, characteristic for each oscillator[7–10] (see also Ref. 11).

With such background we thus came to reconsider the one-dimensional model of a radiant cavity quoted above, investigating it by means of numerical computations the results of which are reported in the present paper. The result found is that a transition to stochasticity occurs here too in a way analogous to that by now familiar for mechanical systems. Moreover a trend toward equipartition in the sense of Boltzmann and Jeans was also observed; this on the one hand confirms the result of Guarneri and Toscani, and on the other hand shows that a heat bath is not necessarily required for such an effect.

c. In Section 2 the model is described and the equations of motion are conveniently handled. Moreover, a discussion is made of the parameters entering the problem, in order to guarantee the conditions which allow the system to be considered as constituted of oscillators weakly coupled by the charged plane. In Section 3 the numerical results obtained are illustrated, and the conclusions follow.

2. THE MODEL

Let us write down the equations of motion for the model quoted in the Introduction, referring to Refs. 2–4 for more details.

Taking an orthogonal frame of reference such that the charged plane lies on the yz coordinate plane and moves along the z axis, and choosing the Coulomb gauge for the field, the unknowns of the problem reduce to the z component $A_z(x,t)$ of the vector potential and to the vertical coordinate $z(t)$ of a suitable point of the charged plane. The equations of motion and boundary conditions are then

$$\begin{gathered} \frac{\partial^2 A_z}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} = -\frac{4\pi}{c}\sigma\delta(x)\dot{z} \\ m\ddot{z} = -\frac{\sigma}{c}\frac{\partial A_z}{\partial t}(0,t) + F(z) \\ A_z(-l,t) = A_z(l,t) = 0 \end{gathered} \tag{1}$$

Here, c is the velocity of light, m and σ are the mass and charge density, respectively, of the charged plane, $2l$ is the distance between the mirrors, while $\delta(\ )$ denotes the Dirac "function," and $F(z)$ represents a mechanical restoring force acting on the plane: precisely, $(1/m)F(z) = -\omega_0^2 z - \alpha z^3$, ω_0 and α being parameters.

The problem of physical interest is to study the distribution of energy among the normal modes of the electromagnetic field, which are considered to be weakly coupled by the charged plane. To this end the normal-mode amplitudes $a_n(t)$ $(n = 1, 2, 3, \ldots)$ are defined by $A_z(x,t) = \sum_{n=1}^{\infty} a_n(t)\, u_n(x)$, where $u_n(x) = 2c(\pi/l)^{1/2}\cos\omega_n x/c$, and $\omega_n = (\pi c/2l)n$.

The model turns then out to be described in these variables by the

Lagrangian function

$$\mathfrak{L}(a,\dot{a},z,\dot{z}) = \frac{1}{2}\sum_{n=1}^{\infty}{}'\left(\dot{a}_n^2 - \omega_n^2 a_n^2\right) + \frac{1}{2}m\dot{z}^2 - V(z) + 2\left(\frac{\pi}{l}\right)^{1/2}\sigma\sum_{n=1}^{\infty}{}' a_n\dot{z}$$

$$V(z) = m\left(\frac{1}{2}\omega_0^2 z^2 + \frac{\alpha}{4}z^4\right) \tag{2}$$

where $a = (a_1, a_3, a_5, \ldots)$, $\dot{a} = (\dot{a}_1, \dot{a}_3, \dot{a}_5, \ldots)$ and $\sum'$ denotes a sum restricted to the odd positive integers $n = 1, 3, 5, \ldots$; indeed one easily sees that the even modes have a trivial free dynamics and can actually be neglected, so that one remains with the problem of the distribution of energy among the odd modes. This is a conservative system, whose total energy is

$$E = \frac{1}{2}m\dot{z}^2 + V(z) + \sum_{n=1}^{\infty}{}' E_n \tag{3}$$

where

$$E_n = \frac{1}{2}\left(\dot{a}_n^2 + \omega_n^2 a_n^2\right), \qquad \omega_n = \frac{\pi c}{2l}n \tag{4}$$

are the energies and the angular frequencies of the normal modes of the field.

The problem is now to handle with a computer the infinite system of equations deduced from (2), namely,

$$\ddot{a}_n + \omega_n^2 a_n = 2\left(\frac{\pi}{l}\right)^{1/2}\sigma\dot{z} \qquad (n = 1, 3, 5, \ldots) \tag{5a}$$

$$\ddot{z} = -2\left(\frac{\pi}{l}\right)^{1/2}\frac{\sigma}{m}\sum_{n=1}^{\infty}{}'\dot{a}_n - \left(\omega_0^2 z + \alpha z^3\right) \tag{5b}$$

As a matter of fact, this system can be rewritten in the form

$$\ddot{a}_n + \omega_n^2 a_n = 2\left(\frac{\pi}{l}\right)^{1/2}\sigma\dot{z} \qquad (n = 1, 3, 5, \ldots) \tag{6a}$$

$$\ddot{z} = -2\left(\frac{\pi}{l}\right)^{1/2}\frac{\sigma}{m}\sum_{n=1}^{\infty}{}'\dot{a}_n^0(t) - \frac{2\pi\sigma^2}{mc}\left[\dot{z} + 2\sum_{k=1}^{N(t)}(-1)^k\dot{z}\left(t - 2\frac{kl}{c}\right)\right] - \left(\omega_0^2 z + \alpha z^3\right) \tag{6b}$$

where $N(t)$ is the integer part of $ct/2l$ and $a_n^0(t) = a_n^0\cos\omega_n t + (\dot{a}_n^0/\omega_n)\sin\omega_n t$ are the free field solutions of (5a), i.e., the solutions with $\sigma = 0$, with initial data $a_n^0, \dot{a}_n^0$. So one sees that Eq. (6b) for the charged plane contains no explicit reference to the variables of the field, apart from the known ones $a_n^0(t)$; instead, it contains explicitly the velocity $\dot{z}$ of the plane at suitable previous times. Such an equation can thus be handled numerically, and the solution so computed can then be inserted into the right-hand side of (6a) as a known term. In such way one can compute $a_n(t)$ and thus the normal-mode energies $E_n(t)$ for any $n = 1, 3, 5, \ldots$.

As to the deduction of Eq. (6b) from the system (5), one first obtains from (5a) an integral expression for $\dot{a}_n(t)$ as a function of $z(t)$ and substitutes it into (5b), exploiting then the identity

$$\sum_{n=1}^{\infty}{}' \int_{-a}^{a} f(x) \cos \frac{\mu\pi x}{2a}\, dx = af(0)$$

for any $a > 0$. Alternatively, one can make use of Kirchhoff's formula for the retarded potentials of the electromagnetic field, applied directly to system (1).

Let us finally come to a discussion of the parameters entering the model, namely, c, l, σ, m, ω_0, α, in order to extract from them dimensionless parameters of particular relevance. In fact, at variance with Refs. 3–4, we always considered the simpler case with $\omega_0 = 0$, so that such parameter will be neglected.

A first relevant dimensionless parameter, already considered in the previous works[2-4] on this model is

$$\gamma = \frac{2\pi l \sigma^2}{mc^2} \tag{7}$$

which can be taken as characterizing the strength of the coupling in the system, as explained in a moment. Indeed the free field has proper frequencies $\omega_n = (\pi c/2l)n$ with a fixed spacing $\Omega = (\pi c/2l)$. In the case of a linear coupling (i.e., with $\alpha = 0$), as shown in Ref. 2, the coupled field has instead proper frequencies ω_n^* with a shift $\delta\omega_n \equiv \omega_n^* - \omega_n$ satisfying the equation

$$\cot(\delta\omega_n/\Omega) = \frac{1}{\gamma}\left(\frac{\delta\omega_n}{\Omega} + n\frac{\pi}{2}\right)$$

This equation has the approximate solutions $\delta\omega_n/\Omega \sim \gamma/n$ if $\gamma/n \ll 1$, and thus the frequency shift $\delta\omega_n$ for all modes is small (with respect to the spacing Ω) provided $\gamma \ll 1$. In this sense we say that, for $\gamma \ll 1$, the field can be considered as constituted of free modes with a weak coupling, measured by γ. This is also confirmed by the remark that γ appears as the ratio of two characteristic times, precisely $mc/2\pi\sigma^2$, which according to (6b) characterizes a damping on the plane due to the coupling with the field, and l/c, which is a typical macroscopic time of the free field.

We come now to a characterization of the nonlinearity. The nonlinearity in the force $F(z)$ acting in the charged plane is determined by α; a more appropriate parameter turns out instead to be the product $E\alpha$, where E is the total energy of the system. Indeed it is easy to show that there is an isomorphism between a dynamical system (2) with a pair (E,α) and another one with a pair (E',α') if $E'\alpha' = E\alpha$, in the sense that the orbits of the one are deduced from the orbits of the other by a change of scales of the z coordinate and of the field amplitude. As a corresponding dimensionless number characterizing the nonlinearity in the model we took

$$\epsilon = \frac{1}{\gamma}\frac{l}{c}\left(\frac{E\alpha}{m}\right)^{1/4} \tag{8}$$

Indeed one easily checks that such parameter admits of a clear physical interpretation, as it turns out to give the ratio of the two maximal accelerations of the charged plane, $\ddot{z} = -\alpha z^3$ and $\ddot{z} = -\gamma(c/l)\dot{z}$, due to the nonlinear mechanical restoring force and to the electromagnetic force [see Eq. (6b)], respectively, at a fixed energy E, if the whole energy is given to the charged plane.

In conclusion, as significant dimensionless parameters in the model we have γ, given by (7), characterizing the strength of the (linear) coupling among the field modes, and ϵ, given by (8), characterizing the nonlinearity of the system. This situation is rather atypical with respect to most familiar systems of interacting particles, where the energy E determines simultaneously both the nonlinearity and the coupling, so that the corresponding integrable systems are obtained by letting E tend to zero. In the present model, instead, an integrable system is obtained either by taking the limit $\epsilon \to 0$, i.e., for example, $E \to 0$ (linearly coupled integrable system), or by taking the limit $\gamma \to 0$ (nonlinear integrable system).

3. DESCRIPTION OF THE NUMERICAL RESULTS

3.1. Numerical Integration

As already anticipated, the system of differential equations for our model as written in the form (6) is particularly suited for numerical integration.[3,4] Indeed, in each time interval $k < ct/2l < k+1$ ($k = 1,2,3,\ldots$) Eq. (6b) for the unknown $z(t)$ takes the form

$$\ddot{z} = -\alpha z^3 - \beta\dot{z} + G(t) + S_k(t) \tag{9}$$

where α and β are constants, while $G(t)$ and $S_k(t)$ are known functions; Eq. (9) can be integrated by any standard numerical scheme. In turn, the function $z(t)$ thus determined, when inserted into each of the equations (6a) for the unknown $a_n(t)$ ($n = 1,3,5,\ldots$), allows one to compute as many of the field amplitudes $a_n(t)$ as one likes.

A few details on the numerical scheme actually used in this paper are described in the Appendix. The important remark should however be made that a discretization of time necessarily implies a cutoff on the frequencies of the modes that can actually be handled. Thus, in our computations, using a time step of typically $5 \times 10^{-3} l/c$, we never considered modes with $n > 25$, i.e., with a proper period less than $0.16 l/c$.

3.2. Stochasticity Thresholds

A first series of experiments was intended to determine the possible existence of stochasticity thresholds analogous to those recently observed in other models of weakly coupled oscillators.[7–11] In such models one proceeded in the following way: all of the energy was given initially to just one mode or to a group of modes of close frequencies, and for each group one looked for the existence of a characteristic energy threshold, in the sense that the energy sharing with other modes was sensible only above that threshold.

To proceed in an analogous way, we took a fixed value of the coupling γ (namely, $\gamma = 0.2\pi$) and considered groups of two nearby modes, for example, the group of modes 1 and 3 (i.e., with $n = 1$ and $n = 3$), or the group of modes 5 and 7 and so on. In each experiment, having fixed the group and a value of the nonlinearity ϵ, the initial condition was defined by assigning 95% of the energy to the modes (in the form of kinetic energy, and equally distributed between them) and the remaining 5% of the energy to the charged plane (again in the form of kinetic energy). The equations of motion were then solved numerically for a time T with typically $T = 10^4 l/c$. The intensity of the energy sharing was measured by the quantity

$$\lambda = \frac{E^{\max} - E^{\min}}{E^{\max}} \tag{10}$$

where $E^{\max} = \max_{0 \leqslant t \leqslant T} E(t)$, $E^{\min} = \min_{0 \leqslant t \leqslant T} E(t)$, $E(t)$ being the sum of the energies of the two initially excited modes at time t. One has obviously $0 \leqslant \lambda \leqslant 1$, with $\lambda = 0$ for no energy sharing, and $\lambda = 1$ when $E^{\min} = 0$, i.e., if at least at one time up to T the considered oscillators had no energy. One can thus draw a curve of λ vs. ϵ for each group.

The results are shown in Fig. 1, where λ is plotted versus ϵ (for fixed $\gamma = 0.2\pi$ and T up to $10^4 l/c$) for the groups (1, 3), (5, 7), (7, 9), (9, 11), and (13, 15). The curves indicate for each group a very sharp transition from essentially no energy sharing to an almost complete energy sharing, at a well-defined critical value ϵ^c of ϵ. In agreement with the results found for several systems of mechanical oscillators,[7–11] the critical value ϵ^c appears to be an increasing function of the frequency of the modes. This is shown in Fig. 2, where ϵ^c, as taken from Fig. 1, is plotted versus the average frequency of the modes of the considered group.

A complete analysis of the model would require a study of the dependence of the thresholds on the coupling parameter γ. As an example, this has been done for the group of modes (5, 7), and the results thus

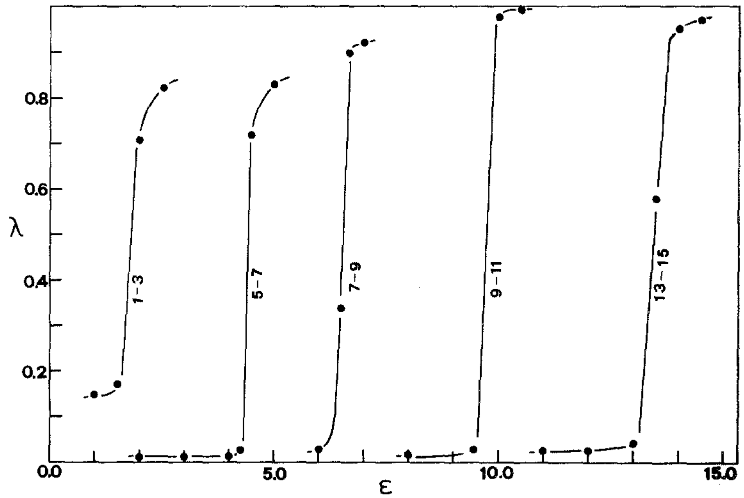

Fig. 1. Intensity of energy sharing, measured by λ [Eq. (10)], versus the nonlinearity parameter ϵ [Eq. (8)] for various groups of modes, as indicated for each curve. For each group, a critical value ϵ^c is rather well exhibited. The coupling γ [Eq. (7)] has the fixed value 0.2π.

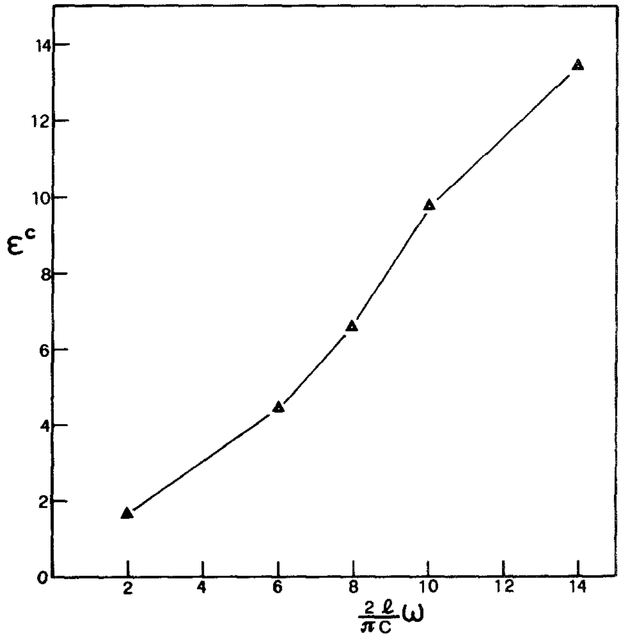

Fig. 2. Critical value ϵ^c for a group of modes (as taken from Fig. 1) versus the average frequency of the modes of the group.

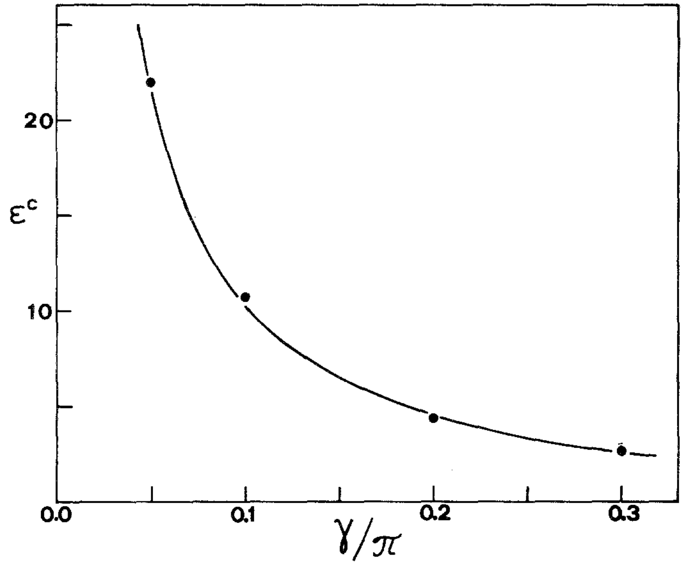

Fig. 3. Critical value ϵ^c for the group of modes (5, 7) versus the coupling parameter γ. The interpolating curve $\epsilon^c = 1.2(\pi/\gamma - 1)$ is also exhibited.

obtained for ϵ^c vs. γ are shown in Fig. 3. As expected, ϵ^c is a decreasing function of γ. In fact, if one makes a corresponding plot of ϵ^c vs. $1/\gamma$, the four points available turn out to be surprisingly well fitted by the straight line $\epsilon^c = 1.2(\pi/\gamma - 1)$, according to which one extrapolates a vanishing value of ϵ^c for $\gamma \approx \pi$. The interpolating curve is also drawn in Fig. 3.

In conclusion, the results reported in Fig. 1 in connection with the energy sharing certainly give a striking evidence of the occurrence of a kind of transition at the threshold ϵ^c. In fact, this is not enough to give complete information on the partition of energy among the modes, because one only knows that above threshold some other modes, in addition to the ones initially excited, are involved in intense energy sharing. Thus this point deserves a further investigation, on which we will report in Section 3.4. Nevertheless, one is certainly authorized to say that, by going through any such threshold ϵ^c, one passes from an ordered motion to a chaotic (or stochastic) motion, as illustrated in the following section.

3.3. Exhibition of Stochasticity Through the Maximal Ljapunov Exponent

It has by now become a quite accepted fact that stochasticity in dynamical systems consists in the rapid, exponential-like, divergence of nearby orbits,[12] also called "sensitive dependence on initial conditions",[13] which can be formalized by means of the notion of the Ljapunov characteristic exponents (LCE) of a system.[14-16] Indeed, one says a dynamical system presents a stochastic behavior in a region of its phase space if its maximal LCE is thereby positive. For an elementary introduction oriented toward numerical applications of the present kind see, for example, Ref. 16.

Just in order to fix the notations, let us recall the definition of the LCEs of a dynamical system described by a differential equation $\dot{x} = f(x)$ in the finite dimensional Euclidean space $\mathbb{R}^n$. Given a solution $x(t)$ with $x(0) = x_0$, one defines then the LCE χ by

$$\chi(x_0, \xi_0) = \lim_{t\to\infty} \frac{1}{t} \ln \|\xi(t)\| \tag{11}$$

where $\xi(t)$ is the solution of the corresponding variational equation with initial condition $\xi(0) = \xi_0$. Having fixed x_0, χ can take at most n different values when ξ_0 is varied in $\mathbb{R}^n$; in fact, however, it takes the maximal one, which we denote by $\chi(x_0)$, apart from a set of initial data ξ_0 of vanishing Lebesgue measure, which is negligible in numerical computations. In practice, having fixed x_0 and chosen any ξ_0, one computes $\xi(t)$ and thus the quantity

$$\chi_t = \frac{1}{t} \ln \|\xi(t)\| \tag{12}$$

looking for its asymptotic behavior when $t \to \infty$. Experience shows that log–log plots are particularly suited to this end.

In the case at hand one is confronted with the problem that the theory of the LCEs has not yet been developed for infinite-dimensional systems. Thus we proceeded by formal analogy and found results which appear to be rather consistent.

First of all we remark that, in the same way as from system (5) one obtains system (6), one also can put the corresponding variational equations of system (6) in the form

$$\ddot{\xi}_n + \omega_n^2 \xi_n = 2\left(\frac{\pi}{l}\right)^{1/2} \dot{\xi}_z \qquad (n = 1, 3, 5, \ldots) \tag{13a}$$

$$\begin{aligned} \ddot{\xi}_z = & -2\left(\frac{\pi}{l}\right)^{1/2} \frac{\sigma}{m} \sum_{n=1}^{\infty}{}' \dot{\xi}_n^0 \\ & - \beta\left[\dot{\xi}_2 + 2\sum_{k=1}^{N(t)} (-1)^k \dot{\xi}_z\left(t - 2\frac{kl}{c}\right)\right] - \left(\omega_0^2 \xi_z + 3\alpha z^2 \xi_z\right) \end{aligned} \tag{13b}$$

where ξ_z and ξ_n are the variations of z and a_n, respectively. Thus one is able to compute as many components of the infinite-dimensional vector $\xi = (\xi_z, \dot{\xi}_z, \xi_1, \dot{\xi}_1, \xi_3, \dot{\xi}_3, \ldots)$ as one likes; in such way one can take as an approximation to $\|\xi(t)\|$ the Euclidean norm in the subspace of the components $(\xi_z, \dot{\xi}_z, \xi_1, \dot{\xi}_1, \ldots, \xi_N, \dot{\xi}_N)$, with an arbitrary N. This procedure turned out to be consistent because, already for small values of t (say, $t > 100l/c$), χ_t was found to be almost completely insensitive to the choice of N; indeed, no change in the first few digits of χ_t was observed by considering several cases of N in the range $0 \leqslant N \leqslant 20$. Thus in practice one can even take $N = 0$, i.e., one can consider only the components $\xi_z, \dot{\xi}_z$ corresponding to the coordinates z and $\dot{z}$ of the charged plane.

Having recalled these technical facts, we come now to an illustration of the results. The aim is to check whether, by going through any of the thresholds ϵ^c defined in terms of the parameter λ as in Fig. 1, one indeed passes from a situation of ordered motions (namely, with $\chi = 0$) to a situation of stochastic motions (namely, with $\chi > 0$), where $\chi = \lim_{t\to\infty} \chi_t$. To this end we considered, for example, two initial conditions with excitation of the group of modes 5 and 7 (in the sense explained in Section 3.2) at $\epsilon = 3.5$ and 5.0, respectively; recall that, according to Fig. 1, one has $\epsilon^c \approx 4.5$. As one sees in Fig. 4, χ_t appears to tend to zero in the first case, and to a well-stabilized positive value in the second case. In this way we seem to be authorized to interpret the thresholds defined by the results of Fig. 1 as stochasticity thresholds.

3.4. Energy Distribution

Thus we have shown that, for a fixed value of the coupling parameter γ, the model presents a stochasticity threshold ϵ^c for each group of oscillators; this can be thought of as a function $\epsilon^c(\omega)$ of the average frequency ω of the group and is indeed an increasing function of ω. As a consequence, there is in fact no problem of energy distribution when only one group of oscillators is initially excited, below threshold.

A sharp change occurs instead when a group is initially excited above its threshold. In such case indeed we know already, from the results of Section 3.2, that there has been at least a moment in which the considered group gave out essentially all of its initial energy to some other modes or to the charged plane (which can be likened to a mode of vanishing frequency).

The problem is then to know how the energy is distributed among all the modes in time average. So we look at the quantities

$$\bar{E}_n(T) = \frac{1}{T}\int_0^T E_n(t)\,dt \tag{14}$$

where $E_n(t)$ is the energy of the nth mode at time t.

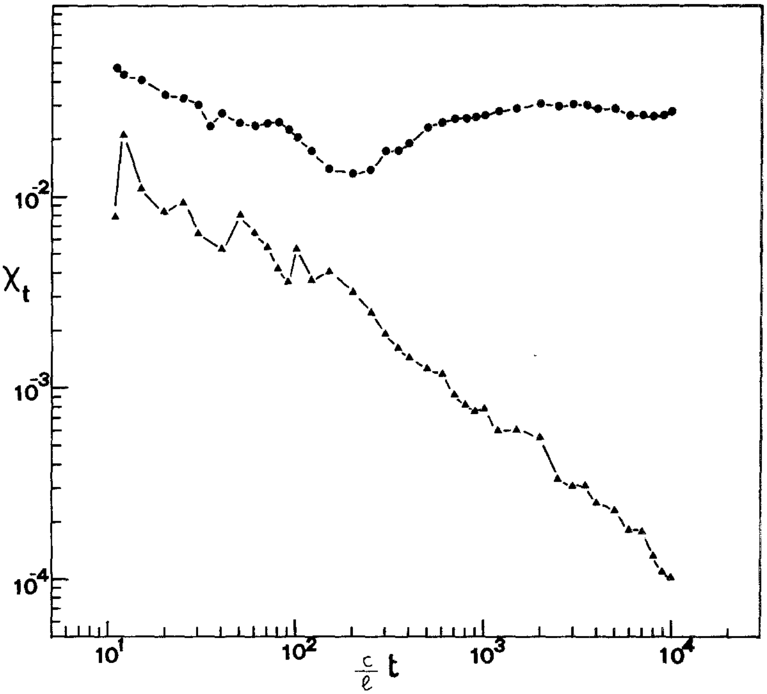

Fig. 4. Exhibiting the transition to stochasticity by means of the maximal Ljapunov characteristic exponent $\chi = \lim_{t\to\infty} \chi_t$ [Eq. (12)]. Here χ_t is plotted versus t in log–log scale for two orbits with initial excitation of the group of modes (5, 7). The first orbit, with $\epsilon = 3.5 < \epsilon^c \approx 4.5$, gives $\chi_t \to 0$, while the second one, with $\epsilon = 5.0 > \epsilon^c$, gives $\chi_t \to a > 0$. In both cases, $\gamma = 0.2\pi$.

We come now to the illustration of a typical result in this connection, with initial conditions of the same type as in Sections 3.2 and 3.3 (95% of the energy to a particular group of modes and 5% the charged plane). Taking for example an initial excitation of the group of modes 5 and 7 with ϵ above threshold, precisely $\epsilon = 7.0$, we made a computation up to time $T = 10^5 l/c$; the spectra [i.e., the time averages (14) versus frequency] at times $10^2, 10^4, 10^5 l/c$ are reported in Fig. 5a. As one sees, the modes of low frequency become rapidly excited, while those of higher frequency require increasingly longer times.

Let us now come to a more detailed analysis. First of all, for what concerns the spectrum at $T = 10^5 l/c$, one may notice that it is quite well approximated by a *plateau* for $0 \leqslant n \leqslant 11$ (equipartition of energy among the modes of low frequency), followed by an exponentially decreasing queue. This corresponds to the pair of straight lines shown in Fig. 6, drawn

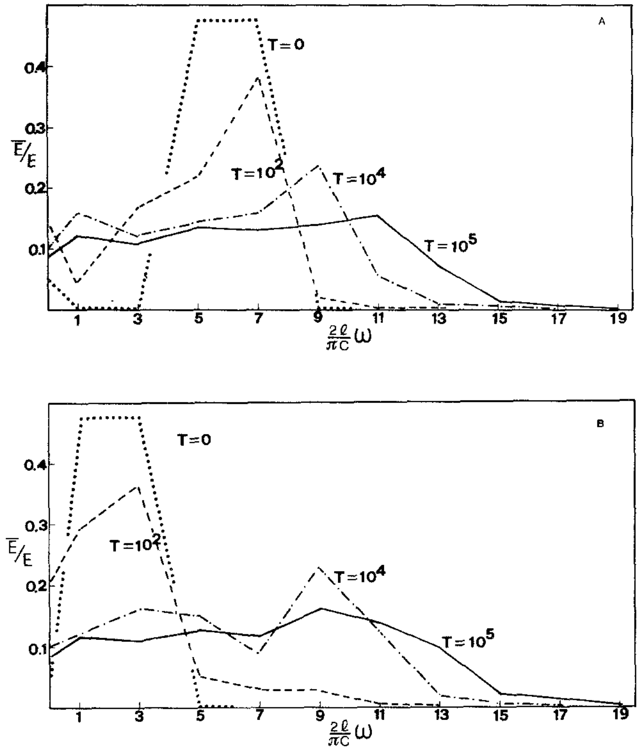

Fig. 5. Energy distribution (in time average) $\overline{E}_n(T)/E$ (E being the total energy) among the modes of the field at times $10^2, 10^4, 10^5 l/c$ for initial excitation of a group of modes above threshold: (a) group (5, 7) at $\epsilon = 7.0$; (b) group (1, 3) at the same value $\epsilon = 7.0$. In both cases, $\gamma = 0.2\pi$.

as an interpolation to the experimental data for $T = 10^5 l/c$ plotted in a semilogarithmic scale.

We are then confronted with two problems, namely, (1) how the situation depends on the class of initial conditions and (2) what can one guess for $T \to \infty$. As to the first problem, we considered an initial condition with excitation of the group of modes 1 and 3 above threshold, again for $\epsilon = 7.0$ and $\gamma = 0.2\pi$. The results shown in Fig. 5b indicate, by comparison with those of Fig. 5a, that the memory of the initial condition is rather rapidly lost.

For what concerns the problem of the limit distribution when $T \to \infty$, already the results of Fig. 5 seem to indicate a trend towards equipartition, the modes of higher frequency requiring, however, longer and longer times to get involved in the process of energy sharing. This latter fact is better

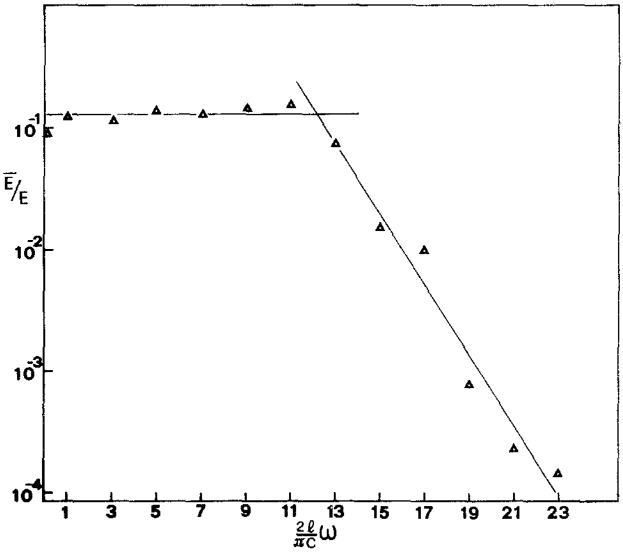

Fig. 6. The data of Fig. 5a for $T = 10^5 l/c$ plotted in semilogarithmic scale, in order to exhibit the interpolation by a *plateau* followed by an exponentially decreasing queue.

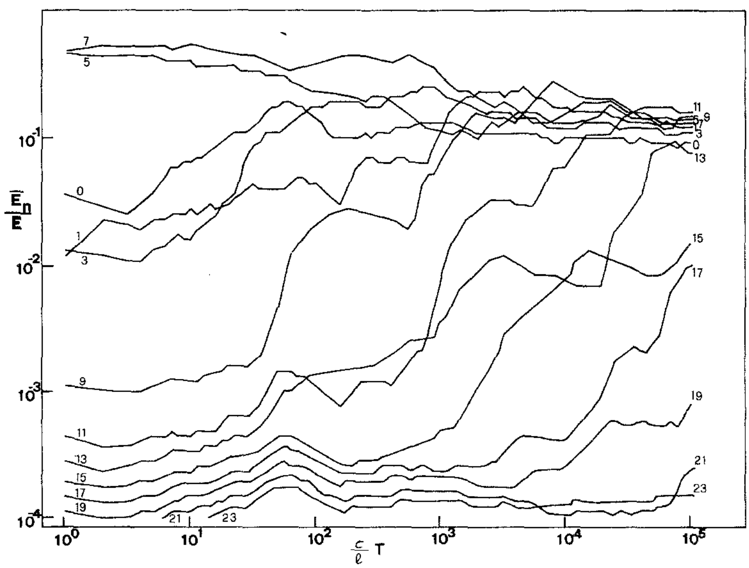

Fig. 7. Time averages $\bar{E}_n(T)$ divided by total energy E vs. T for $n = 0, 1, 3, 5, \ldots, 23$ and the same orbit of Fig. 5a, up to $T = 10^5 l/c$ in log–log scale, exhibiting how each mode starts sharing energy at a characteristic time, increasing with the frequency.

visualized in Fig. 7, where the time averages $\bar{E}_n(T)$ for all modes up to $n = 23$ are plotted versus T up to $T = 10^5 l/c$ in log–log scale, for the same orbit considered in Fig. 5a. As one sees, the general trend for the higher modes to jump into the main "sharing group," each at a characteristic time, is well exhibited.

4. CONCLUSIONS

Let us first point out the main results exhibited by our computations. In our opinion the two most relevant points are the following ones.

(a) There exist stochasticity thresholds. For initial conditions below such thresholds there are essentially no energy exchanges among the modes of the field and the motions are of ordered type, in the sense that the maximal LCE vanishes; for initial conditions above such thresholds there are intense energy exchanges involving some modes, and the motions are chaotic, the maximal LCE being positive.

(b) Concerning the energy exchanges for initial conditions above threshold, there seems to be a trend toward equipartition (in time average). However, the modes of higher frequencies appear to require longer and longer times in order to be involved in the energy exchanges.

For what concerns point (a), the existence of such thresholds is in agreement with the results already familiar to us for the mechanical models studied in Refs. 7–11. For what concerns point (b), the trend toward equipartition with increasingly longer times for the higher frequencies is in agreement, apart from the existence of a threshold for such effect, with the analytical results obtained by Guarneri and Toscani.[5] The latter results concern a model analogous to the present one, but with a purely linear spring and the superposition of a stochastic perturbation on the charged plane, simulating a thermal contact with a heat reservoir. More in general, let us recall that the existence of a characteristic time, depending on frequency, for an oscillator to be involved in the energy exchanges leading to equipartition, is indeed a feature particularly stressed by Boltzmann and Jeans.[6] Our results can then be summarized by saying that they indeed indicate a trend toward equipartition, precisely in the sense of Boltzmann and Jeans, but just for initial conditions above threshold.

Thus one is confronted with a dynamical situation which partially agrees with the usually expected one of equipartition but is, in a sense, paradoxical: the simultaneous presence in a model of both equipartition and stochasticity thresholds is not easily understandable. Work in this direction is still in progress. Whether this is a general feature or not is in our opinion a very interesting open problem. We only would like to recall in this connection that such a simultaneous presence was explicitly envisaged by Nernst[17,18] in 1916.

APPENDIX

We shortly describe here the numerical algorithm we used to integrate Eqs. (6b) and (13b). These are both of the form

$$\ddot{x} = -\beta\dot{x} + f(x,t) \tag{A1}$$

where β is a parameter and f a known function. Having fixed a time step h, one can write

$$x(t+h) = x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t) + \frac{h^3}{6}\dddot{x}(t) + O(h^4)$$

$$x(t-h) = x(t) - h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t) - \frac{h^3}{6}\dddot{x}(t) + O(h^4)$$

from which one immediately gets

$$x(t+h) + x(t-h) = 2x(t) + h^2\ddot{x}(t) + O(h^4) \tag{A2a}$$

$$\dot{x}(t) = \frac{x(t+h) - x(t-h)}{2h} + O(h^2) \tag{A2b}$$

Substituting into (A1) $\dot{x}$ and $\ddot{x}$ as obtained from (A2), one thus gets

$$\left(1 + \frac{\beta h}{2}\right)x(t+h) = 2x(t) - \left(1 - \frac{\beta h}{2}\right)x(t-h) + h^2 f(x(t),t) + O(h^4)$$

which defines the approximation scheme, when the term $O(h^4)$ is neglected.

In practice, it is more suitable to introduce the quantities $x_j = x(jh)$ and $D_j = x_{j+1} - x_j$, so that one gets the recursive relations

$$\left.\begin{aligned} x_{j+1} &= x_j + D_j \\ D_{j+1} &= \left[C^- D_j + h^2 f(x_j, jh)\right]/C^+ \end{aligned}\right\} \qquad (j = 0, 1, 2, \ldots)$$

with $C^{\pm} = 1 \pm \beta h/2$. By the initial conditions the initial point x_0 is known, while D_0 is obtained by means of a Taylor expansion at third order. In such a way the method is correct at any step only up to third order in the time step; its speediness makes it, however, particularly suited for long computations.

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